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Algebraic structure of the Lie algebra $so(2,1)$ for a quantized field in a vibrating cavity

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Abstract
It is shown that the generators of the Lie algebra $so(2,1)$ can be defined in terms of annihilation and creation operators of infinite boson modes while an angular momentum algebra can be introduced with finite boson modes. The $so(2,1)$ algebraic structure is shown to exist in the quantized field in a vibrating cavity, which considerably simplifies the studies on the system's eigenvalue problem and the dynamical evolution.

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1. Introduction

It is well known [1–5] that introducing an angular momentum algebra often simplifies greatly the studies of the eigenvalue problem and dynamics of a system composed of finite boson modes. Such examples are the two-mode description of quantum interference of condensates [1], the three-mode description of two-photon Raman processes in a microcavity [2] and of energy spectra and dynamics of a multi-component Bose–Einstein condensate [3], as well as the description of four-wave mixing with matter waves [4]. Recently, we have investigated two-photon Raman processes [5] and the general structure of Boson–Einstein condensates [6] with arbitrary but finite boson modes by introducing an angular momentum algebra. In all the above-mentioned examples, three introduced components J_α ($\alpha = x, y, z$) of an angular momentum $\boldsymbol{J}$ are expressed as a bilinear form of creation and annihilation operators of the finite boson modes. These three components are obviously the infinitesimal generators of the Lie group $SO(3)$ (or $SU(2)$) [7] and hence describe the algebraic structure of that group. Most recently, Coleman *et al* [8] have developed a supersymmetric representation of spin operators which unifies the Schwinger and Abrikosov representations of $SU(N)$ spin operators, allowing a second-quantized treatment of representations of the $SU(N)$ group with both symmetric and antisymmetric character. At this time, it is natural to

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ask whether there exist other algebraic structures instead of the angular momentum algebraic structure for a system having finite or infinite boson modes, which will be the subject of this paper.

In this paper we shall show that an angular momentum algebra can always be introduced for arbitrary but finite photon modes while infinite boson modes can result in a totally different algebraic structure characterized by the three infinitesimal generators of the Lie algebra $so(2,1)$. We shall discuss their applications to some physical systems, and pay much attention to the problem of the quantized field in a one-dimensional vibrating cavity. It is shown that the introduced algebraic structure of Lie algebra $so(2,1)$ for the quantized field in a vibrating cavity greatly simplifies the corresponding eigenvalue problem and dynamical evolution of the quantized field. This paper is organized as follows. In section 2, we outline the main spirit of the angular momentum algebra introduced by bilinear combinations of the creation and annihilation operators of finite boson modes, and briefly discuss its applications. In section 3, we introduce the algebraic structure of Lie algebra $so(2,1)$ by defining its three infinitesimal generators in terms of the creation and annihilation operators of infinite boson modes. In section 4, we apply the results in section 3 to discussing the eigenvalue problem of the quantized field in a vibrating cavity, and section 5 concludes the paper.

2. Angular momentum algebra for finite boson modes

Suppose there exist $2f+1$ boson modes where f is finite and can take any positive half-integer value $f = 1/2, 3/2, 5/2, \ldots$ (even boson modes) and integers $f = 1, 2, \ldots$ (odd boson modes). Introducing

$$J_- = J_+^\dagger = \sum_{j=-f}^{f} \sqrt{(f-j)(f+j+1)}\, a_j^\dagger a_{j+1} \tag{1a}$$

$$J_z = \sum_{j=-f}^{f} j a_j^\dagger a_j \qquad J^2 = J_- J_+ + J_z^2 + J_z \tag{1b}$$

it is easily shown that they define an angular momentum $\boldsymbol{J}$ with its three components $J_x = (J_+ + J_-)/2$, $J_y = \mathrm{i}(J_- - J_+)/2$ (i.e. $J_\pm = J_x \pm \mathrm{i}J_y$) and J_z since

$$[J_+, J_-] = 2J_z \qquad [J_\pm, J_z] = \mp J_\pm \tag{2a}$$

$$[J_\alpha, J_\beta] = \mathrm{i}\epsilon_{\alpha\beta\gamma} J_\gamma \qquad \alpha, \beta, \gamma = x, y, z. \tag{2b}$$

Hence we have shown that for arbitrary but finite boson modes, we can always introduce an angular momentum $\boldsymbol{J}$ in terms of creation and annihilation operators of the $2f+1$ boson modes. J_z denotes its z-component, J_- (J_+) represents the corresponding lowering (raising) operator and the Casimir operator J^2 is nothing but the squared angular momentum.

Before ending discussions of the angular momentum algebra for arbitrary but finite boson modes, we emphasize that in the case of even boson modes, half-integer subscripts for the creation and annihilation operators are introduced only for convenience so that we can describe both even and odd modes in a united way. The notation of half-integer subscripts can be avoided in the case of even modes by defining $J_- = J_+^\dagger = \sum_{j=-n}^{n} \sqrt{(n-j)(n+j+1)}\, a_j^\dagger a_{j+1}$ and $J_z = \sum_{j=-n}^{n} (j-1/2) a_j^\dagger a_j$ with a positive integer n so that all the subscripts $j = 0, \pm 1, \pm 2, \ldots, \pm n$ are integers now. Similarly minus subscripts can also be avoided by translating all of them by a certain integer, e.x., defining $J_- = \sum_{j=1}^{2n+1} \sqrt{j(2n+1-j)}\, a_j^\dagger a_{j+1}$ and $J_z = \sum_{j=1}^{2n+1} (j-n-3/2) a_j^\dagger a_j$ for even boson modes, as well as $J_- = \sum_{j=1}^{2f+1} \sqrt{j(2f+1-j)}\, a_j^\dagger a_{j+1}$ and $J_z = \sum_{j=1}^{2f+1} (j-f-1) a_j^\dagger a_j$ for odd boson modes.

Let us now mention some special previously studied situations of the general expression (1) of the angular momentum in terms of creation and annihilation operators of arbitrary but finite boson modes. Equation (1) for $f = 1/2$ is easily shown to give, except for different subscript notation, the definitions of two angular momenta $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ introduced in [4] for investigating four-wave mixing with matter waves. Obviously, equation (1) for $f = 1$ reduces to the connection between field variables with an angular momentum for the three-mode description of multi-component Bose–Einstein condensates [3]. Such a connection has been shown to greatly simplify discussions of these problems [3,4]. As for its applications to cavity quantum electrodynamics problems, let us mention the following quite general Hamiltonian [5]:

$$H = \tfrac{1}{2}\delta\omega\sigma_z + \tilde{\omega}N + \Omega J_z + g(J_-\sigma_{21} + J_+\sigma_{12}) \tag{3}$$

where the meanings of the constants δ, $\tilde{\omega}$, Ω and g are referred to [5]. This Hamiltonian describes a series of multi-coupled-channel cavity QED models, for instance, the models in the multi-Λ configuration with arbitrary ($2f = 1, 2, 3, \ldots$) Λ channels, and can be easily diagonalized [5]. For $f = 1/2$, this Hamiltonian, together with the angular momentum definition (1), describes a two-wave system in either Λ or Ξ configurations [9], and it becomes a three-wave system describing two-photon Raman processes with $a_0 \equiv a_{\mathrm{P}}$, $a_{-1} \equiv a_{\mathrm{S}}$ and $a_1 \equiv a_{\mathrm{A}}$ describing the pump, Stokes and anti-Stokes modes respectively [2].

3. $so(2, 1)$ algebraic structure for infinite boson modes

In this case, we define three Hermitian operators $A_\alpha = A_\alpha^\dagger$ ($\alpha = x, y, z$) by the relations

$$A_x = \tfrac{1}{2}\sum_{j=1}^{\infty}\sqrt{j(j+1)}(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) \tag{4a}$$

$$A_y = \frac{\mathrm{i}}{2}\sum_{j=1}^{\infty}\sqrt{j(j+1)}(a_j^\dagger a_{j+1} - a_{j+1}^\dagger a_j) \tag{4b}$$

$$A_z = \sum_{j=1}^{\infty} j a_j^\dagger a_j. \tag{4c}$$

It is easily shown that

$$[A_x, A_y] = -\mathrm{i}A_z \tag{5a}$$

$$[A_x, A_z] = -\mathrm{i}A_y \qquad [A_y, A_z] = \mathrm{i}A_x \tag{5b}$$

$$[A_+, A_-] = -2A_z \qquad [A_\pm, A_z] = \mp A_\pm \tag{6}$$

where $A_\pm = A_x \pm \mathrm{i}A_y$ or

$$A_- = A_+^\dagger = \sum_{j=1}^{\infty}\sqrt{j(j+1)}a_j^\dagger a_{j+1}. \tag{7}$$

Note that the summation upper limit ∞ in the definitions of equation (4) manifests that infinite boson modes are needed to construct the Lie algebra $so(2, 1)$. Obviously the commutation relations (5) for the three operators A_α ($\alpha = x, y, z$) are identical to those for the three infinitesimal generators of the Lie algebra $so(2, 1)$ [7] and hence describe the algebraic structure of the Lie algebra $so(2, 1)$. The corresponding Casimir in this case is $C = A_z^2 - (A_x^2 + A_y^2) = A_z^2 - (A_+A_- + A_-A_+)/2$, satisfying the relations

$$[C, A_\alpha] = 0 \qquad \alpha = x, y, z. \tag{8}$$

Utilizing the expressions of $A_\pm$ and A_z, and after some manipulations, we can put the Casimir operator into the form

$$C = \sum_{j,k=1}^{\infty} a_j^\dagger \left(jk a_k^\dagger a_j - \sqrt{jk(j+1)(k+1)} a_{k+1}^\dagger a_{j+1} \right) a_k. \tag{9}$$

It is well known that there exist four different unitary irreducible representations for the Lie algebra $so(2, 1)$ in general [7]. However, only one irreducible representation corresponding to the one having a lower bound with no upper bound [7] suits for the case considered here due to the particular form of A_z in equation (4c). Let us illustrate this point. Suppose $|\lambda, m\rangle$ denote the common eigenket of operators C and A_z with eigenvalues λ and m respectively, i.e. $C|\lambda, m\rangle = \lambda|\lambda, m\rangle$ and $A_z|\lambda, m\rangle = m|\lambda, m\rangle$. From equation (4$c$), one realizes that the minimum eigenvalue for A_z is the total boson number N, which is the eigenvalue of the operator $N = \sum_{j=1}^{\infty} a_j^\dagger a_j$ (throughout this paper, we use the same symbol N to denote both the total boson number operator and its eigenvalues for simplicity), and we therefore have the relation $A_-|\lambda, m = N\rangle = 0$. Utilizing this relation and the relation $A_+A_- = A_z^2 - A_z - C$, one easily obtains $\lambda = N(N-1)$, i.e. the Casimir operator C can only take one value $N(N-1)$ for a fixed total boson number. The above discussion explains the fact that only one irreducible representation corresponding to the one having a lower bound with no upper bound suits for our case, and the eigenvalues of the operator A_z are $m = N, N+1, N+2, \ldots$ [7]. In addition we can, by utilizing equation (5), easily obtain the useful transformation formula

$$\exp(-\mathrm{i}\theta A_y) A_z \exp(\mathrm{i}\theta A_y) = A_z \cosh\theta + A_x \sinh\theta. \tag{10}$$

We have proved the conclusion that infinite boson modes can introduce an algebraic structure of Lie algebra $so(2, 1)$. It is emphasized that the commutation relations (5) for the operators A_α differ from equation (2) for the three angular momentum components J_α ($\alpha = x, y, z$) and therefore the two operator sets (J_x, J_y, J_z) and (A_x, A_y, A_z) describing the cases involving finite and infinite boson modes respectively, represent different algebraic structures.

Before ending this section, it is pointed out that we can in fact introduce many independent $so(2, 1)$ Lie algebras from infinite boson modes $(a_j, a_j^\dagger, j = 1, 2, 3, \ldots)$. To see this, we define $B_-(n, k) = B_+^\dagger(n, k)$, $B_\pm(n, k) = B_x(n, k) \pm \mathrm{i}B_y(n, k)$ and

$$B_-(n, k) = \sum_{j=0}^{\infty} \sqrt{\left(j + \frac{k}{n}\right)\left(j + 1 + \frac{k}{n}\right)} a_{nj+k}^\dagger a_{n(j+1)+k} \tag{11a}$$

$$B_z(n, k) = \sum_{j=0}^{\infty} \left(j + \frac{k}{n}\right) a_{nj+k}^\dagger a_{nj+k} \tag{11b}$$

where n is a fixed positive integer, and $k = 1, 2, \ldots, n$. Note that $A_\alpha \equiv B_\alpha(n = 1, k = 1)$, $\alpha = x, y, z, +, -$. It is then easily shown that they satisfy the commutation relations,

$$[B_+(n, k), B_-(n, k)] = -2B_z(n, k) \tag{12a}$$

$$[B_\pm(n, k), B_z(n, k)] = \mp B_\pm(n, k) \tag{12b}$$

where $B_\pm(n, k) = B_x(n, k) \pm \mathrm{i}B_y(n, k)$. Therefore three operators $\boldsymbol{B}(n, k) = (B_x(n, k), B_y(n, k), B_z(n, k))$ for fixed positive integers n and k give the three generators of Lie algebra $so(2, 1)$. The corresponding Casimir in this case is $C_{n,k} = B_z^2(n, k) - [B_x^2(n, k) + B_y^2(n, k)]$, satisfying the relations

$$[C_{n,k}, B_\alpha(n, k)] = 0 \qquad \alpha = x, y, z. \tag{13}$$

The Casimir operator can be put into the form

$$C_{n,k} = \frac{k}{2n}\left(\frac{k}{n} - 1\right) a_k^\dagger a_k + \sum_{j,m=0}^{\infty} g_{j,m}(k/n) a_{nj+k}^\dagger a_{nm+k}^\dagger a_{nj+k} a_{nm+k}$$
$$- \sum_{j,m=0}^{\infty} f_{j,m}(k/n) a_{nj+k}^\dagger a_{n(m+1)+k}^\dagger a_{n(j+1)+k} a_{nm+k} \tag{14}$$

where $g_{j,m}(k/n) = (j + k/n)(m + k/n)$ and $f_{j,m}(k/n) = \sqrt{g_{j,m}(k/n) g_{j+1,m+1}(k/n)}$. It is easily seen that the Casimir operator $C_{n,k}$ for $n = k = 1$ is nothing but the Casimir operator C in equation (9). Similarly we can derive the following useful transformation formulas:

$$\exp(-\mathrm{i}\theta B_y(n,k)) B_z(n,k) \exp(\mathrm{i}\theta B_y(n,k)) = B_z(n,k)\cosh\theta + B_x(n,k)\sinh\theta \tag{15}$$

where n is a fixed positive integer, and $k = 1, 2, \ldots, n$. It is pointed out that operators $\boldsymbol{B}(n,k)$ and operators $\boldsymbol{B}(n,k')$ for $k \neq k'$ are two setS of independent generators since $[B_\alpha(n,k), B_\beta(n,k')] = 0$ $(\alpha, \beta = x, y, z)$ if $k \neq k'$.

4. $so(2, 1)$ structure of the quantized field in a vibrating cavity

In this section, we apply the results in the last section to study the quantized field in a one-dimensional cavity formed by two perfectly reflecting mirrors with one mirror fixed at the position $x = 0$ and the other allowed to oscillate according to the relation $x = L\exp[q_0\cos(n\omega t/L]$. Here $\omega = \pi/L$ is the fundamental eigenfrequency of the cavity with a fixed length L, and $n = 1, 2, \ldots$ describe the nth harmonic resonance case where the moving mirror oscillates with the nth unperturbed eigenfrequency $n\omega$ [10,11]. Under the rotating-wave approximation, the quantized field in such a vibrating cavity is shown [11] to be described by the Hamiltonian $H = H' + H^{(p)}$ with $H^{(p)}$ denoting the parametric part and

$$\frac{H'}{\Omega} = \sum_{j=1}^{\infty} j a_j^\dagger a_j + \epsilon \sum_{j=1}^{\infty} \sqrt{j(j+n)} (a_j^\dagger a_{j+n} + a_{j+n}^\dagger a_j) \tag{16}$$

where Ω and ϵ are two constants [11], and positive integer n $(= 1, 2, \ldots)$ describes the nth harmonic resonance case. We shall study the fundamental resonance case $(n = 1)$ and harmonic resonance cases $(n \geqslant 2)$ separately, beginning with the former case.

The fundamental resonance case $n = 1$. In this case, the parametric part $H^{(p)}$ is absent [11], so the total Hamiltonian $H \equiv H'$. Utilizing equations (4) and (10), it is easy to put the total Hamiltonian H into the form

$$\frac{H}{\Omega} = A_z + 2\epsilon A_x = \sqrt{1 - (2\epsilon)^2} \exp(-\mathrm{i}\theta A_y) A_z \exp(\mathrm{i}\theta A_y) \tag{17}$$

where $\theta = \tanh^{-1}(2\epsilon)$. Consequently, the total Hamiltonian H can be unitarily transformed into the diagonal form

$$\exp(\mathrm{i}\theta A_y) H \exp(-\mathrm{i}\theta A_y) = \bar{\Omega} \sum_{j=1}^{\infty} j a_j^\dagger a_j \tag{18}$$

where $\bar{\Omega} = \Omega\sqrt{1 - (2\epsilon)^2}$. Therefore all the energy eigenvalues are of the form $\bar{\Omega}\sum_{j=1}^{\infty} j n_j$ corresponding to the energy eigenket $\exp(\mathrm{i}\theta A_y)|n_1, n_2, \ldots, n_j, \ldots\rangle$. Here $|n_1, n_2, \ldots, n_j, \ldots\rangle$ is the eigenket of photon number operators $a_j^\dagger a_j$, $j = 1, 2, \ldots$ with the eigenvalue n_j (=0, 1, 2, ...). It is seen that exploring the algebraic structure of Lie algebra $so(2, 1)$ for infinite photon modes provides us with a very simple method to diagonalize the

Hamiltonian describing the quantized field in a vibrating cavity in the fundamental resonance case. Apart from the relatively simple nature compared to the previous diagonalization method developed in [11], the present method permits us to show easily from equations (8) and (17) that the Casimir operator C in equation (9) is a conserved quantity since $[H, C] = 0$ and its value is $N(N-1)$ according to the discussions given between equations (9) and (10). Here N is the initial total photon number of the system. It is worthwhile to point out that such a conserved quantity (the Casimir operator C) and its value seem to be hard to discover without exploring the $so(2, 1)$ algebraic structure of the quantized field in a vibrating cavity.

The harmonic resonance cases $n \geqslant 2$. In these cases, the parametric part $H^{(p)}$ is nonzero [11]. However, once the Hamiltonian H' has been diagonalized, the diagonalization of the total Hamiltonian in the presence of the parametric part can be treated with the help of the method in [11]. We therefore focus on the diagonalization of the Hamiltonian H' in this paper. The Hamiltonian H' in equation (16) can expressed as $H' = \Omega\sum_{k=1}^{n} H(n, k)$ with [11]

$$H(n,k) = \sum_{j=0}^{\infty}(nj+k)a_{nj+k}^{\dagger}a_{nj+k} + \epsilon\sum_{j=0}^{\infty}\sqrt{(nj+k)(nj+n+k)}(a_{nj+k}^{\dagger}a_{n(j+1)+k} + \mathrm{hc}) \tag{19}$$

where hc denotes Hermitian conjugation. It is seen from equations (11) and (15) that $H(n, k)$ can be put into a more concise form

$$H(n,k) = nB_z(n,k) + 2n\epsilon B_x(n,k) = n\sqrt{1-(2\epsilon)^2}U_k(\theta)B_z(n,k)U_k^{\dagger}(\theta) \tag{20}$$

where the unitary operator $U_k(\theta) = \exp(-\mathrm{i}\theta B_y(n,k))$, and $\theta = \tanh^{-1}(2\epsilon)$. Defining a total unitary operator $U(\theta) = \prod_{k=1}^{n} U_k(\theta)$, noting $U_k(\theta)U_{k'}(\theta) = U_{k'}(\theta)U_k(\theta)$ (due to $[B_y(n,k), B_y(n,k')] = 0$) and $U^{\dagger}(\theta)H(n,k)U(\theta) = U_k^{\dagger}(\theta)H(n,k)U_k(\theta)$ (due to $H(n,k)U_{k'}(\theta) = U_{k'}(\theta)H(n,k)$ for $k \neq k'$ because $U_{k'}(\theta)$ and $H(n,k)$ for $k \neq k'$ do not share any common annihilation and creation operators of the photon modes), we then have

$$U^{\dagger}(\theta)H'U(\theta) = \bar{\Omega}\sum_{k=1}^{n} nB_z(n,k) = \bar{\Omega}\sum_{j=1}^{\infty} ja_j^{\dagger}a_j \tag{21}$$

where $\bar{\Omega} = \Omega\sqrt{1-(2\epsilon)^2}$. Once again, it is seen that exploring the $so(2, 1)$ algebraic structure for infinite photon modes also provides us with a very simple method to diagonalize the Hamiltonian describing the quantized field in a vibrating cavity in harmonic resonance cases in the absence of the parametric part $H^{(p)}$. In dealing with the diagonalization of the Hamiltonian H' in these cases, the present method is obviously much simpler than the previous method also developed by us [11].

Let us present some further discussions related to the harmonic resonance cases. First of all, it has been shown in the last section that, for infinite boson modes, we can introduce the $so(2, 1)$ algebraic structure in several different ways. However, one particular way may not be helpful for the diagonalization of a specific model. For instance, the $so(2, 1)$ algebraic structure described by the generators A_j, $j = x, y, z$ in equation (4) is very useful in diagonalizing the Hamiltonian in the fundamental resonance case but is not so in harmonic cases. In other words, for any specific model, one needs to sort out the most suitable way to introduce the $so(2, 1)$ algebraic structure. Secondly, if H' in equation (16) is the total Hamiltonian for some systems, then the energy eigenvalues for such systems are of the form $\bar{\Omega}\sum_{j=1}^{\infty} jn_j$ corresponding to the energy eigenket $U^{\dagger}(\theta)|n_1, n_2, \ldots, n_j, \ldots\rangle$ with $U^{\dagger}(\theta) = \prod_{k=1}^{n}\exp(\mathrm{i}\theta B_y(n,k)) = \exp[\mathrm{i}\theta\sum_{k=1}^{n} B_y(n,k)]$. Here we have made use of the property $[B_y(n,k), B_y(n,k')] = 0$. Note that the expressions of energy eigenvalues and eigenkets here are also suitable for the fundamental resonance case. Thirdly, for systems whose total Hamiltonian is H' given by equation (16), it is easily seen from equations (13) and (20) that all the n Casimir operators $C_{n,k}$, $k = 1, 2, \ldots, n$ given by equation (14) are commutative with H' and hence all of them are conserved quantities. In addition, it can be shown by the similar arguments presented between equations (9) and (10) that each Casimir operator $C_{n,k}$ takes a fixed value $\bar{N}_k(\bar{N}_k - 1)$ where $\bar{N}_k = kN_k/n$ and $N_k = \sum_{j=0}^{\infty} n_{nj+k}$ is the initial total photon number of all the photon modes $a_{nj+k}^{\dagger}a_{nj+k}$, $j = 0, 1, 2, \ldots$. Again we emphasize that these conserved quantities $C_{n,k}$, $k = 1, 2, \ldots, n$ and their values are hard to discover without exploring the $so(2, 1)$ algebraic structure. Finally, we point out that studies of the dynamical evolution of such systems are simple matters since we have diagonalized the corresponding Hamiltonian, and we therefore omit discussions on the dynamics of such systems in this paper.

5. Conclusions

In summary, we have shown that an angular momentum algebra can always be introduced for arbitrary but finite photon modes while infinite boson modes can result in a totally different algebraic structure characterized by the three infinitesimal generators of Lie algebra $so(2, 1)$. We have shown that the introduced $so(2, 1)$ algebraic structure for the quantized field in a vibrating cavity greatly simplifies the corresponding eigenvalue problem and dynamical evolution of the quantized field in a vibrating cavity. Besides, exploring algebraic structures of systems composed of finite and infinite boson modes may be a powerful way in finding systems' conserved quantities just as the cases discussed in the last section. It is well known that introducing an angular momentum algebra often greatly simplifies the studies of the eigenvalue problem and dynamics of a system. Therefore it is our hope that the $so(2, 1)$ algebraic structure for infinite boson modes might also find wide applications to other systems besides the system of the quantized field in a vibrating cavity discussed here. It is pointed out that only the algebraic structure rather than the complete group $SO(2, 1)$ is needed for the purposes of this paper and therefore we have only mentioned Lie algebra $so(2, 1)$ rather than Lie group $SO(2, 1)$ [12].

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